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Entire solutions of certain type of differential equations

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Abstract

By utilizing Nevanlinna's value distribution theory of meromorphic functions, we solve the transcendental entire solutions of the following type of nonlinear differential equations in the complex plane:

$$f^n(z) + P(f) = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z},$$

where p_1 and p_2 are two small functions of e^z, and α_1, α_2 are two nonzero constants with some additional conditions, and $P(f)$ denotes a differential polynomial in f and its derivatives (with small functions of f as the coefficients) of degree no greater than $n-1$.

Keywords: Differential equation; Transcendental entire solution; Nevanlinna theory

1. Introduction and results

Let $\mathbb{C}$ denote the complex plane and $f(z)$ a nonconstant meromorphic function on $\mathbb{C}$. In 1925, R. Nevanlinna derived and developed the value distribution theory with the well-known Jensen formula as the starting point. The theory mainly consists of the so-called first and second fundamental theorems, expressed in terms of three quantities $T(r, f)$, $m(r, f)$ and $N(r, f)$ associated with a given meromorphic function f; they are called characteristic function, proximate function and counting function of f, respectively. Throughout the paper, $S(r, f)$ will be used to denoted any quantity that satisfies $S(r, f) = o(1)T(r, f)$ as $r \to \infty$, outside possibly an exceptional set of r values of finite linear measure. We shall call a meromorphic function $a(z)$ a small function of $f(z)$ if $T(r, a) = S(r, f)$. By the lemma on the logarithmic derivative of meromorphic function, we have the estimation: $m(r, f^{(k)}/f) = S(r, f)$, which plays a very important role in the studies of the growth of property of meromorphic functions, especially on meromorphic solutions of differential equations in complex plane. A differential polynomial $P(f)$ in f is a polynomial in f and its derivatives with small functions of f as the coefficients. The notation $\mathscr{A}$ is defined to be the family of all meromorphic functions which satisfy $\overline{N}(r, 1/h) + \overline{N}(r, h) = S(r, h)$. Note that all functions in family $\mathscr{A}$ are transcendental, and all functions of the form $be^{\lambda z}$ are functions in family $\mathscr{A}$, where λ is any nonzero constant and b is a rational function. We refer the reader to the book [5] for the details of the Nevanlinna theory and its standard notations.

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Nevanlinna's value distribution theory of meromorphic functions has been used to study or tackle the growth, oscillation, solvability and existence of entire or meromorphic solutions of differential equations in complex domains, see, e.g., [4,6]. Some nonlinear differential equations have been studied in [3,9,10]. Specifically, it shows in [10] that the equation $4f^3 + 3f'' = -\sin 3z$ has exactly three nonconstant entire solutions, namely $f_1(z) = \sin z$, $f_2(z) = \frac{\sqrt{3}}{2}\cos z - \frac{1}{2}\sin z$, and $f_3(z) = -\frac{\sqrt{3}}{2}\cos z - \frac{1}{2}\sin z$. Furthermore, in [7] and [10] the following more general results are obtained, respectively.

Theorem A. *Let $n \geqslant 4$ be an integer, and $P(f)$ denote an algebraic differential polynomial in f of degree $\leqslant n-3$. Let p_1, p_2 be two nonzero polynomials, α_1 and α_2 be two nonzero constants with $\alpha_1/\alpha_2 \neq$ rational. Then the differential equation*

$$f^n(z) + P(f) = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z}$$

has no transcendental entire solutions.

Theorem B. *Let $n \geqslant 3$ be an integer, $P(f)$ be an algebraic differential polynomial in f of degree $\leqslant n-3$, $b(z)$ be a meromorphic function, and λ, c_1, c_2 be three nonzero constants. Then the differential equation*

$$f^n(z) + P(f) = b(z)\left(c_1 e^{\lambda z} + c_2 e^{-\lambda z}\right)$$

has no transcendental entire solutions $f(z)$, that satisfy $T(r, b) = S(r, f)$.

It is conjectured in [7] that the conclusion in Theorem A remains true when the degree of the differential polynomial $P(f)$ is $n-2$ or $n-1$. In the present paper, we prove the following results which are improvement or complementarity of Theorems A and B.

Theorem 1. *Let $n \geqslant 2$ be a positive integer. Let f be a transcendental entire function, $P(f)$ be a differential polynomial in f of degree $\leqslant n-1$. If*

$$f^n(z) + P(f) = p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z}, \tag{1.1}$$

where p_i $(i = 1, 2)$ are nonvanishing small functions of e^z, α_i $(i = 1, 2)$ are positive numbers satisfying $(n-1)\alpha_2 \geqslant n\alpha_1 > 0$, then there exists a small function γ of f such that

$$(f - \gamma)^n = p_2 e^{\alpha_2 z}. \tag{1.2}$$

Theorem 2. *Let $n \geqslant 2$ be positive integers, α_i $(i = 1, 2)$ be real numbers and $\alpha_1 < 0 < \alpha_2$. Let p_1, p_2 be small functions of e^z. If there exists a transcendental entire function f satisfying the differential equation* (1.1)*, where $P(f)$ is a differential polynomial in f of degree not exceeding $n-2$, then $\alpha_1 + \alpha_2 = 0$, and there exist constants c_1, c_2 and small functions β_1, β_2 with respect to f such that*

$$f = c_1\beta_1 e^{\alpha_1 z/n} + c_2\beta_2 e^{\alpha_2 z/n}. \tag{1.3}$$

Moreover, $\beta_i^n = p_i$, $i = 1, 2$.

Remark. It is easy to construct some examples to show that if the degree of $P(f)$ is $n-1$, then the solutions of Eq. (1.1) may not be the form in (1.3).

Corollary 1. *The differential equation*

$$f^3 + f + 4f'' = 2\cos 3z$$

has exactly three entire solutions: $f_1(z) = 2\cos z$, $f_2(z) = -\frac{1}{2}\cos z - \frac{\sqrt{3}}{2}\sin z$, and $f_3(z) = -\frac{1}{2}\cos z + \frac{\sqrt{3}}{2}\sin z$.

Theorem 3. *Suppose that $n \geqslant 2$ is a positive integer, p_i $(i = 1, 2)$ are small functions of e^z, and α_i $(i = 1, 2)$ are positive numbers satisfying $(n-1)\alpha_2 \geqslant n\alpha_1 > 0$. If α_1/α_2 is irrational, then the differential equation* (1.1) *has no entire solutions, where $P(f)$ is a differential polynomial in f of degree $\leqslant n-1$.*

2. Some lemmas

The following lemmas will be used in the proofs of the theorems.

Lemma 1. *(See Clunie's lemma [1,2].) Suppose that $f(z)$ is meromorphic and transcendental in the plane and that*

$$f^n(z)P(f)=Q(f),$$

where $P(f)$ and $Q(f)$ are differential polynomials in f with functions of small proximity related to f as the coefficients and the degree of $Q(f)$ is at most n. Then

$$m\big(r,P(f)\big)=S(r,f).$$

Lemma 2. *(See [5].) Suppose that f is a nonconstant meromorphic function and $F=f^n+Q(f)$, where $Q(f)$ is a differential polynomial in f with degree $\leqslant n-1$. If $N(r,f)+N(r,1/F)=S(r,f)$, then*

$$F=(f+\gamma)^n,$$

whereby γ is meromorphic and $T(r,\gamma)=S(r,f)$.

Lemma 3. *(See [8].) Suppose that h is a function in family $\mathscr{A}$. Let $f=a_0h^p+a_1h^{p-1}+\cdots+a_p$ and $g=b_0h^q+b_1h^{q-1}+\cdots+b_q$ be polynomials in h with all coefficients being small functions of h and $a_0b_0a_p\not\equiv 0$. If $q\leqslant p$, then $m(r,g/f)=S(r,h)$.*

3. Proof of Theorem 1

First of all, we write $P_{n-1}(f)$ as the following:

$$P(f)=\sum_{j=0}^{n-1}b_jM_j(f), \tag{3.1}$$

where b_j are small functions of f, $M_0(f)=1$, $M_j(f)$ $(j=1,2,\dots,n-1)$ are differential monomials in f of degree j. Without loss of generality, we assume that $b_0\not\equiv 0$, otherwise, we do the transformation $f=f_1+c$ for a suitable constant c. From (1.1), we have

$$\frac{1}{p_1e^{\alpha_1z}+p_2e^{\alpha_2z}-b_0}+\sum_{j=1}^{n-1}\frac{b_j}{p_1e^{\alpha_1z}+p_2e^{\alpha_2z}-b_0}\frac{M_j(f)}{f^j}\left(\frac{1}{f}\right)^{n-j}=\left(\frac{1}{f}\right)^n. \tag{3.2}$$

Note that $m(r,M_j(f)/f^j)=S(r,f)$, and by Lemma 3 we have

$$m\left(r,\frac{1}{p_1e^{\alpha_1z}+p_2e^{\alpha_2z}-b_0}\right)=S\big(r,p_1e^{\alpha_1z}+p_2e^{\alpha_2z}\big)=S(r,f).$$

Therefore, the left-hand side of (3.2) is a polynomial in $1/f$ of degree at most $n-1$ with coefficients being small proximate functions of $1/f$. Hence

$$m\left(r,\frac{1}{f}\right)=S(r,f). \tag{3.3}$$

Taking the derivatives in both sides of (1.1) gives

$$nf^{n-1}f'+\big(P(f)\big)'=\big(p_1'+\alpha_1p_1\big)e^{\alpha_1z}+\big(p_2'+\alpha_2p_2\big)e^{\alpha_2z}. \tag{3.4}$$

By eliminating e^{α_2z} and e^{α_1z}, respectively from (1.1) and the above equation, we get

$$\big(p_2'+\alpha_2p_2\big)f^n-p_2nf^{n-1}f'+\big(p_2'+\alpha_2p_2\big)P(f)-p_2\big(P(f)\big)'=\beta e^{\alpha_1z}, \tag{3.5}$$

$$\big(p_1'+\alpha_1p_1\big)f^n-p_1nf^{n-1}f'+\big(p_1'+\alpha_1p_1\big)P(f)-p_1\big(P(f)\big)'=-\beta e^{\alpha_2z}, \tag{3.6}$$

where $\beta = p_1 p_2' - p_2 p_1' + (\alpha_2 - \alpha_1) p_1 p_2$ which is a small function of f. We note that β cannot vanish identically, otherwise, by integration we get $e^{(\alpha_2-\alpha_1)z} = Cp_1/p_2$ for a constant C, which is impossible. From (3.5) and (3.6), we get

$$m\big(r, e^{\alpha_j z}\big) \leqslant nT(r,f) + S(r,f), \quad j = 1, 2. \tag{3.7}$$

On the other hand, from (1.1), we have

$$nT(r,f) = m\big(r, f^n\big) = m\big(r, f^n + P(f)\big) \leqslant T\big(r, p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z}\big) + S(r,f). \tag{3.8}$$

Therefore, $S(r, e^{\alpha_1 z}) = S(r, e^{\alpha_2 z}) = S(r,f) := S(r)$. From (3.2), we have

$$\frac{e^{\alpha_i z}}{p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z} - b_0} + \sum_{j=1}^{n-1} \frac{b_j e^{\alpha_i z}}{p_1 e^{\alpha_1 z} + p_2 e^{\alpha_2 z} - b_0} \frac{M_j}{f^j} \left(\frac{1}{f}\right)^{n-j} = \frac{e^{\alpha_i z}}{f^n}, \quad i = 1, 2.$$

It follows that

$$m\left(r, \frac{e^{\alpha_i z}}{f^n}\right) = S(r), \quad i = 1, 2. \tag{3.9}$$

In the following, we prove

$$m\left(r, \frac{e^{\alpha_1 z}}{f^{n-1}}\right) = S(r). \tag{3.10}$$

For a fixed $r > 0$, let $z = re^{i\theta}$. The interval $[0, 2\pi)$ can be expressed as the union of the following three disjoint sets:

$$E_1 = \left\{\theta \in [0, 2\pi) \;\Big|\; \frac{|f(z)|}{|e^{(\alpha_2-\alpha_1)z}|} \leqslant 1\right\},$$
$$E_2 = \left\{\theta \in [0, 2\pi) \;\Big|\; \frac{|f(z)|}{|e^{(\alpha_2-\alpha_1)z}|} > 1,\ \big|e^z\big| \leqslant 1\right\},$$
$$E_3 = \left\{\theta \in [0, 2\pi) \;\Big|\; \frac{|f(z)|}{|e^{(\alpha_2-\alpha_1)z}|} > 1,\ \big|e^z\big| > 1\right\}.$$

By the definition of the proximate function, we have

$$m\left(r, \frac{e^{\alpha_i z}}{f^{n-1}}\right) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left|\frac{e^{\alpha_i z}}{f^{n-1}(z)}\right| d\theta = I_1 + I_2 + I_3,$$

where

$$I_i = \frac{1}{2\pi} \int_{E_i} \log^+ \left|\frac{e^{\alpha_i z}}{f^{n-1}(z)}\right| d\theta, \quad i = 1, 2, 3.$$

For $\theta \in E_1$, we have $|f(z)| \leqslant |e^{(\alpha_2-\alpha_1)z}|$. Since $\frac{e^{\alpha_1 z}}{f^{n-1}(z)} = \frac{e^{\alpha_2 z}}{f^n(z)} \frac{f(z)}{e^{(\alpha_2-\alpha_1)z}}$, we get

$$I_1 \leqslant m\left(r, \frac{e^{\alpha_2 z}}{f^n}\right) = S(r).$$

For $\theta \in E_2$, we have $|e^{\alpha_1 z}| \leqslant 1$, and thus $|\frac{e^{\alpha_1 z}}{f^{n-1}(z)}| \leqslant \frac{1}{f^{n-1}(z)}$. It follows from (3.3) that

$$I_2 \leqslant m\left(r, \frac{1}{f^{n-1}}\right) = S(r).$$

For $\theta \in E_3$, we have $|f(z)| > |e^{(\alpha_2-\alpha_1)z}|$. Therefore,

$$\left|\frac{e^{\alpha_1 z}}{f^{n-1}(z)}\right| \leqslant \frac{|e^{\alpha_1 z}|}{|e^{(n-1)(\alpha_2-\alpha_1)z}|} = \frac{1}{|e^{(n-1)\alpha_2 z - n\alpha_1 z}|}.$$

By the assumption $(n-1)\alpha_2 \geqslant n\alpha_1$, we get $|e^{\alpha_1 z}|/|f^{n-1}(z)| \leqslant 1$. Therefore, we have $I_3 = 0$. Hence (3.10) holds.

It follows from (3.5) that

$$f^{n-1}\varphi=\beta\frac{e^{\alpha_1 z}}{f^{n-1}}\cdot f^{n-1}-R(f), \tag{3.11}$$

where $\varphi=(p_2'+\alpha_2 p_2)f-np_2f'$, and

$$R(f)=\big(p_2'+\alpha_2 p_2\big)P(f)-p_2\big(P(f)\big)'$$

which is a differential polynomial in f of degree at most $n-1$. By Lemma 1, we get $m(r,\varphi)=S(r)$. Note that φ is entire, we have $N(r,\varphi)=S(r)$. Hence $T(r,\varphi)=S(r)$, i.e., φ is a small function of f. By the definition of φ, we get

$$f'=\frac{p_2'+\alpha_2 p_2}{np_2}f-\frac{\varphi}{np_2}.$$

Substituting the above equation into (3.6) gives

$$f^n-\frac{np_1\varphi}{\beta}f^{n-1}-\frac{p_2(p_1'+\alpha_1 p_1)}{\beta}P(f)+\frac{p_1p_2}{\beta}\big(P(f)\big)'=p_2e^{\alpha_2 z}.$$

By Lemma 2, we see that there exists a small function γ of f such that $(f-\gamma)^n=p_2e^{\alpha_2 z}$. This also completes the proof of Theorem 1.

4. Proof of Theorem 2

We only discuss the case $\alpha_1+\alpha_2\geqslant 0$. The case $\alpha_1+\alpha_2\leqslant 0$ can be discussed similarly. Suppose that f is a transcendental entire solution of (1.1). Similar to the proof of Theorem 1, we can still get (3.3)–(3.9). For a fixed $r>0$, let $z=re^{i\theta}$. We can express the interval $[0,2\pi)$ as the union of the following three disjoint sets:

$$E_1=\left\{\theta\in[0,2\pi)\;\Big|\;\frac{|f^2(z)|}{|e^{(\alpha_2-\alpha_1)z}|}\leqslant 1\right\},$$
$$E_2=\left\{\theta\in[0,2\pi)\;\Big|\;\frac{|f^2(z)|}{|e^{(\alpha_2-\alpha_1)z}|}>1,\ \left|e^z\right|\leqslant 1\right\},$$
$$E_3=\left\{\theta\in[0,2\pi)\;\Big|\;\frac{|f^2(z)|}{|e^{(\alpha_2-\alpha_1)z}|}>1,\ \left|e^z\right|>1\right\}.$$

By the definition of the proximate function, we have

$$m\left(r,\frac{e^{(\alpha_1+\alpha_2)z}}{f^{2n-2}}\right)=\frac{1}{2\pi}\int_0^{2\pi}\log^+\left|\frac{e^{(\alpha_1+\alpha_2)z}}{f^{2n-2}(z)}\right|d\theta=I_1+I_2+I_3,$$

where

$$I_j=\frac{1}{2\pi}\int_{E_j}\log^+\left|\frac{e^{(\alpha_1+\alpha_2)z}}{f^{2n-2}(z)}\right|d\theta,\quad j=1,2,3.$$

For $\theta\in E_1$, we have

$$\left|\frac{e^{(\alpha_1+\alpha_2)z}}{f^{2n-2}(z)}\right|=\left|\frac{e^{2\alpha_2 z}}{f^{2n}(z)}\cdot\frac{f^2(z)}{e^{(\alpha_2-\alpha_1)z}}\right|\leqslant\left|\frac{e^{\alpha_2 z}}{f^n(z)}\right|^2.$$

Thus by (3.9), we get $I_1\leqslant S(r)$. For $\theta\in E_2$, it follows from $|e^z|\leqslant 1$ and $\alpha_1+\alpha_2\geqslant 0$ that $|e^{(\alpha_1+\alpha_2)z}|\leqslant 1$. Therefore,

$$\left|\frac{e^{(\alpha_1+\alpha_2)z}}{f^{2n-2}(z)}\right|\leqslant\frac{1}{|f^{2n-2}(z)|}.$$

Then by (3.3), we get $I_2\leqslant S(r)$. For $\theta\in E_3$, we have $|f^2(z)|>|e^{(\alpha_2-\alpha_1)z}|$. Thus

$$\left|\frac{e^{(\alpha_1+\alpha_2)z}}{f^{2n-2}(z)}\right| < \left|\frac{e^{(\alpha_1+\alpha_2)z}}{e^{(n-1)(\alpha_2-\alpha_1)z}}\right| = \frac{1}{|e^{((n-2)\alpha_2-n\alpha_1)z}|} \leqslant 1.$$

It follows that $I_3 \leqslant S(r)$. Hence we have

$$m\left(r, \frac{e^{(\alpha_1+\alpha_2)z}}{f^{2n-2}}\right) = S(r, f). \tag{4.1}$$

Multiplying (3.5) by (3.6) gives

$$f^{2n-2}\varphi + Q(f) = -\beta^2 e^{(\alpha_1+\alpha_2)z}, \tag{4.2}$$

where $Q(f)$ is differential polynomial in f of degree at most $2n-2$, and

$$\varphi = \big((p_1' + \alpha_1 p_1)f - np_1 f'\big)\big((p_2' + \alpha_2 p_2)f - np_2 f'\big). \tag{4.3}$$

From (4.2) and by Lemma 1, we get $m(r, \varphi) = S(r, f)$. Therefore, $T(r, \varphi) = S(r, f)$.

If $(p_1' + \alpha_1 p_1)f - np_1 f' \equiv 0$, then by integration we get $f^n = cp_1 e^{\alpha_1 z}$, for a nonzero constant c. Therefore, $f = ae^{\alpha_1 z/n}$ for a small function a of f. Thus we see that the left-hand side of (1.1) is a polynomial in $e^{\alpha_1 z/n}$ of degree n. However, the right-hand side of (1.1) cannot be a polynomial in $e^{\alpha_1 z/n}$. Hence $(p_1' + \alpha_1 p_1)f - np_1 f' \not\equiv 0$. Similarly, we have $(p_2' + \alpha_2 p_2)f - np_2 f' \not\equiv 0$. Therefore, $\varphi \not\equiv 0$.

Let

$$\big(p_2' + \alpha_2 p_2\big)f - np_2 f' = h. \tag{4.4}$$

Then we have

$$\big(p_1' + \alpha_1 p_1\big)f - np_1 f' = \frac{\varphi}{h}. \tag{4.5}$$

By eliminating f' and f, respectively from (4.4) and (4.5), we get

$$f = \frac{p_1}{\beta}h - \frac{\varphi p_2}{\beta}\frac{1}{h} \tag{4.6}$$

and

$$f' = \frac{p_1' + \alpha_1 p_1}{n\beta}h - \frac{(p_2' + \alpha_2 p_2)\varphi}{n\beta}\frac{1}{h}, \tag{4.7}$$

where $\beta = p_1 p_2' - p_2 p_1' + (\alpha_2 - \alpha_1)p_1 p_2$ which is a small function of f, and cannot vanish identically. From (4.6), we see that

$$2T(r, h) = T(r, f) + S(r, f).$$

Therefore, any small function of f is also a small function of h. And from the definition of φ we see that h is a function in family $\mathscr{A}$. Thus h'/h is a small function of f. By taking derivative in both sides of (4.6), we get

$$f' = \left(\left(\frac{p_1}{\beta}\right)' + \frac{p_1}{\beta}\frac{h'}{h}\right)h - \left(\left(\frac{\varphi p_2}{\beta}\right)' - \frac{\varphi p_2}{\beta}\frac{h'}{h}\right)\frac{1}{h}. \tag{4.8}$$

Comparing the coefficients of the right-hand side of (4.7) and (4.8), we deduce that

$$\frac{p_1' + \alpha_1 p_1}{n\beta} = \left(\frac{p_1}{\beta}\right)' + \frac{p_1}{\beta}\frac{h'}{h}, \tag{4.9}$$

$$\frac{(p_2' + \alpha_2 p_2)\varphi}{n\beta} = \left(\frac{\varphi p_2}{\beta}\right)' - \frac{\varphi p_2}{\beta}\frac{h'}{h}. \tag{4.10}$$

By integrating (4.9) and (4.10), respectively, we get

$$p_1 e^{\alpha_1 z} = d_1\left(\frac{p_1}{\beta}h\right)^n, \qquad p_2 e^{\alpha_2 z} = d_2\left(\frac{p_2\varphi}{\beta}\frac{1}{h}\right)^n, \tag{4.11}$$

where d_1 and d_2 are two nonzero constants. From the above two equations, we see that there exist two small functions β_1 and β_2 of e^z such that $p_i = \beta_i^n$, $i = 1, 2$, and

$$p_1 p_2 e^{(\alpha_1+\alpha_2)z} = d_1 d_2 \left(\frac{p_1 p_2 \varphi}{\beta^2}\right)^n. \tag{4.12}$$

The right-hand side of the above equation is a small function of f, and thus a small function of e^z. Therefore, the above equation holds only when $\alpha_1 + \alpha_2 = 0$. Furthermore, from (4.11), we see that there exist two nonzero constants c_1 and c_2 such that

$$\frac{p_1}{\beta} h = c_1 \beta_1 e^{\alpha_1 z/n}, \qquad \frac{p_2 \varphi}{\beta} \frac{1}{h} = -c_2 \beta_2 e^{\alpha_2 z/n}. \tag{4.13}$$

Finally, from (4.6), we get (1.3).

5. Proof of Theorem 3

If f is a transcendental entire solution of (1.1), then by Theorem 1, there exists a small function γ of f such that (1.2) holds. And thus $N(r, 1/(f-\gamma)) = S(r, f)$, i.e., γ is an exceptional small function of f. Eq. (1.2) also shows that there exist two small functions ω_1 and ω_2 of f such that $f' = \omega_1 f + \omega_2$. By substituting this equation into (1.1), we see that $p_1 e^{\alpha_1 z}$ is a polynomial in f of degree $k < n$. By Lemma 2, there exist two small functions a and γ_1 of f such that

$$a(f - \gamma_1)^k = p_1 e^{\alpha_1 z}. \tag{5.1}$$

Therefore, γ_1 is also an exceptional small function of f. Since any transcendental entire function cannot have two exceptional small functions, we deduce that $\gamma_1 = \gamma$. From (1.2) and the above equation, we get

$$e^{(n\alpha_1 - k\alpha_2)z} = \frac{p_2^k a^n}{p_1^n}. \tag{5.2}$$

The right-hand side of the above equation is a small function of f, and thus a small function of e^z. Hence we get $n\alpha_1 - k\alpha_2 = 0$. Therefore, α_1/α_2 must be a rational number, which contradicts the assumption. This also completes the proof of Theorem 3.

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